Non-parametric inference for stationary pairwise interaction point process

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Abstract

Among models, allowing to introduce interaction between points, we find the large class of Gibbs models of spatial point processes coming from statistical physics. Such models can produce repulsive as well as attractive point pattern. In this paper, we focus on the main class of Gibbs models which is the class of pairwise interaction point processes characterized by the Papangelou conditional intensity. We suggest a new non-parametric estimate of the pairwise interaction function in the Papangelou conditional intensity for stationary pairwise interaction point process. An order bound for the bias of the resulting estimator is given. Strong uniform consistency is established by a class of stationary Gibbs random fields and the finite range property.

keywords: Georgii-Nguyen-Zessin formula, kernel estimator non-parametric estimation, pairwise interaction point process, Papangelou conditional intensity, strong uniform consistency, Orlicz spaces

1 Introduction

The theory of spatial models is a growing field over the last decade, with various applications in several domains such as ecology (Diggle [8]), forestry (Matérn [20]), spatial epidemiology (Lawson [18]) and astrophysics (Neyman and Scott [23]). Gibbs point processes arose in statistical physics as models for interacting particle systems. Gibbs point processes in $\mathbb{R}^d$ can be defined and characterized through the Papangelou conditional intensity (Møller and Waagepetersen [21]). It is also an important tool in the context of simulations of Gibbs processes. Examples of Markov and non-Markov

Gibbs point process models and their conditional intensities are presented in Baddeley et al [1], Møller and Waagepetersen ([21], [22]). In this paper, we are concerned with non-parametric statistics for stationary pairwise interaction point processes (a special case of a Gibbs process) which describes the interaction between pairs of points by a function (called a pairwise interaction function). They have been introduced in statistical literature by Ripley and Kelly [31], Daley and Vere-Jones [6] and Georgii [12]. These provide a large variety of complex patterns starting from simple potential functions (or pairwise interaction functions) which are easily interpretable as attractive or repulsive forces acting among points, and they are of practical importance because of their ability to model a wide variety of spatial point patterns, especially those displaying some degree of spatial regularity. Our objective of this work is to study estimating non-parametric interaction function in the Papangelou conditional intensity. We suggest a new non-parametric estimate of the pairwise interaction function in the Papangelou conditional intensity for stationary pairwise interaction point process. Sufficient conditions to strong uniform consistency are obtained for the resulting estimator, by stationary field of dependent random variables. Note also that the main results of this work are obtained via assumptions of belonging to Orlicz spaces induced by exponential Young functions for stationary real random fields. Our results also carry through the most important particular case of Orlicz spaces random fields. Many attempts have been tried to estimate the potential function from point pattern data in a parametric framework: maximization of likelihood approximations (Ogata and Tanemura [26], Ogata and Tanemura [27], Penttinen [29]), pseudolikelihood maximization (Besag et al. [2], Jensen and Møller [15]) and also some ad hoc methods (Strauss [32], Ripley [31], Hanisch and Stoyan [13], Diggle and Gratton [9], Fiksel [10], Takacs [33], Billiot and Goulard [3]).

The rest of the paper is organized as follows. Section 2 sets up the generic notation and the basic tools of point processes in $\mathbb{R}^d$. We present the model for stationary pairwise interaction point process and the assumptions which will be considered in the sequel in Section 3. In Section 4, we present our main results; on both the estimation method and strong uniform consistencies for the resulting estimator. The last section is devoted to the proofs.

2 Generic notation and Basic tools

Let $\mathscr{B}^d$ be the Borel σ-algebra (generated by open sets) in $\mathbb{R}^d$ (the d-dimensional space) and $\mathscr{B}^d_O \subseteq \mathscr{B}^d$ be the system of all bounded Borel sets. A point process $\mathbf{X}$ in $\mathbb{R}^d$ is a locally finite random subset of $\mathbb{R}^d$, i.e. the number of points $N(\Lambda) = n(\mathbf{X}_\Lambda)$ of the restriction of $\mathbf{X}$ to Λ is a finite random variable whenever Λ is a bounded Borel set of $\mathbb{R}^d$ (see Daley and Vere-Jones [6]). We define the space of locally finite point configurations in $\mathbb{R}^d$ as $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x}_\Lambda) < \infty, \forall \Lambda \in \mathscr{B}^d_0\}$, where $\mathbf{x}_\Lambda = \mathbf{x} \cap \Lambda$. We equip N_{lf} with σ-algebra $\mathscr{N}_{lf} = \sigma\{\{\mathbf{x} \in N_{lf} : n(\mathbf{x}_\Lambda) = m\}, m \in \mathbb{N}_0, \Lambda \in \mathscr{B}^d_0\}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\} =$

$\{0,1,2,3,\ldots\}$. The volume of a bounded Borel set Λ of $\mathbb{R}^d$ is denoted by $|\Lambda|$ and $o = (0,\ldots,0)$. Let $||t||$ be the Euclidean norm for a point $t \in \mathbb{R}^d$ and $\sum^{\neq}$ signifies summation over distinct pairs. Let $\mathbf{x}$ is a realization of a Gibbs point process $\mathbf{X}$ in Λ (window). In the context of finite point processes, a point process $\mathbf{X}$ is a Gibbs process if and only if $\mathbf{X}$ has a probability density $f(\mathbf{x})$ with respect to the distribution of the unit rate Poisson point process on Λ, such that if $\mathbf{x},\mathbf{y}$ are two possible configurations with $\mathbf{x} \subset \mathbf{y}$, then $f(\mathbf{y}) > 0$ implies $f(\mathbf{x}) > 0$. The Papangelou conditional intensity of $\mathbf{X}$ can then be calculated as $\lambda(u,\mathbf{x}) = f(\mathbf{x} \cup \{u\})/f(\mathbf{x})$ for any configuration $\mathbf{x}$ and any point $u \in \Lambda$ with $u \notin \mathbf{x}$. The Papangelou conditional intensity can be interpreted as follows: for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$, $\lambda(u,\mathbf{x})\mathrm{d}u$ corresponds to the conditional probability of observing a point in a ball of volume $\mathrm{d}u$ around u given the rest of the point process is $\mathbf{x}$. In general it is not possible to deal with densities of infinite point processes. However, the Papangelou conditional intensity of Gibbs point processes $\mathbf{X}$ in $\mathbb{R}^d$ (Møller and Waagepetersen [21]) is a function $\lambda : \mathbb{R}^d \times N_{lf} \to \mathbb{R}_+$ and characterized by the Georgii-Nguyen-Zessin (GNZ) formula (see Papangelou [28] and Zessin [34] for historical comments and Georgii [11] or Nguyen and Zessin [25] for a general presentation). The Georgii-Nguyen-Zessin (GNZ) formula states that for any non-negative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathrm{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathrm{E} \int_{\mathbb{R}^d} h(u,\mathbf{X})\lambda(u,\mathbf{X})\mathrm{d}u. \tag{2.1}$$

Let $\mathbf{X} \otimes \mathbf{X}$ be the point process on $\mathbb{R}^d \times \mathbb{R}^d$ consisting of all pairs (u,v) of distinct points of $\mathbf{X}$. It follows immediately from the GNZ formula (2.1) that $\mathbf{X} \otimes \mathbf{X}$ is a Gibbs point process with (two-point) Papangelou conditional intensity $\lambda(u,v,\mathbf{x}) = \lambda(u,\mathbf{x})\lambda(v,\mathbf{x} \cup \{u\})$, for $u,v \in \mathbb{R}^d, \mathbf{x} \in N_{lf}$, meaning that the GNZ formula in the form

$$\mathrm{E} \sum_{u,v \in \mathbf{X}}^{\neq} h(u,v,\mathbf{X} \setminus \{u,v\}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{E}\, h(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v \tag{2.2}$$

is satisfied for any non-negative measurable function $h(u,v,\mathbf{x})$ on $\mathbb{R}^d \times \mathbb{R}^d \times N_{lf}$.

3 Model

The pairwise interaction point process is characterized by its conditional intensity Papangelou defined by

$$\lambda(u,\mathbf{x}) = g_0(u) \exp\left(- \sum_{v \in \mathbf{x} \setminus u} g_0(\{u,v\}) \right)$$

and note that $g_0(u,v) = g_0(v,u)$ (i.e. symmetric pairwise interaction).

If we consider a special case where $g_0(u)$ is a constant and $g_0(\{u,v\}) = g(v-u)$ is translation invariant. In this case, the pairwise interaction point process is called stationary.

Throughout this paper, we define a stationary pairwise interaction point process via the Papangelou conditional intensity at a location u given by

$$\lambda_{\beta^\star}(u,\mathbf{x}) = \beta^\star \exp\left(-\sum_{v\in\mathbf{x}\setminus u} g(v-u)\right) \tag{3.1}$$

where $\beta^\star$ is the true value of the Poisson intensity parameter, g represents the non-negative pairwise interaction potential defined on $\mathbb{R}^d$. The pairwise interaction between points may also be described in terms of the pairwise interaction function $G = \exp(-g)$. In this semi-parametric model (3.1), the estimator of the Poisson intensity parameter $\beta^\star$ represented the first step in our procedure in the paper [5], we have established its strong consistency and asymptotic normality, we also considered its finite-sample properties simulation study. Now, we develop a method of non-parametric estimation of the pairwise interaction function G, with the class of model defined by (3.1), under assumptions as general as possible.

The basic assumption throughout this paper is the Papangelou conditional intensity has a finite range R, i.e.

$$\lambda_{\beta^\star}(u,\mathbf{x}) = \lambda_{\beta^\star}(u,\mathbf{x}_{B(u,R)}), \tag{3.2}$$

for any $u \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, where $B(u,R)$ is the closed ball in $\mathbb{R}^d$ with center u and radius R.

4 Main results

Suppose that a single realization $\mathbf{x}$ of a point process $\mathbf{X}$ is observed in a bounded window $\Lambda_n \in \mathscr{B}_O^d$ where $(\Lambda_n)_{n\geq 1}$ is a sequence of cubes growing up to $\mathbb{R}^d$. We face a missing data problem, which in the spatial point process literature is referred to as a problem of edge effects, we can avoid this problem by reducing the window by introducing the $2R$-interior of the cubes Λ_n, i.e.

$$\Lambda_{n,R} = \{u \in \Lambda_n : B(u,2R) \subset \Lambda_n\}$$

and assume this has non-zero area. The choice of $\Lambda_{n,R}$ is induced by the fact that when u is on the edge of $\Lambda_{n,R}$ and $v \in B(u,R)$, then we can observe the point v. We have assumed a known interaction range R. In practice, R is often estimated by maximizing a profile pseudolikelihood over a grid. We assume that the support of the interaction function $G = \exp(-g)$ is $T = \{t \in \mathbb{R}^d; g(t) > 0, \text{ for } ||t|| < R\}$. Throughout in this paper, h is a non-negative measurable function defined for all $w \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$, by

$$h(w,\mathbf{x}) = \mathbb{1}(\mathbf{x} \cap B(w,R) = \emptyset), \tag{4.1}$$

and we also introduce the following function

$$\bar{F}(o,w) = \mathrm{E}[h(o,\mathbf{X})h(w,\mathbf{X})] = \mathbb{P}(\mathbf{X}\cap B(o,R)=\emptyset, \mathbf{X}\cap B(w,R)=\emptyset). \tag{4.2}$$

To estimate the function $\beta^{\star}\bar{F}(o,t)$, we propose an empirical estimator $\widehat{\bar{F}}_n(t)$ defined for $t\in T$ by

$$\widehat{\bar{F}}_n(t) = \frac{1}{|\Lambda_{n,R}|}\sum_{u\in\mathbf{X}_{\Lambda_{n,R}}} h(u,\mathbf{X}\setminus\{u\})h(t+u,\mathbf{X}\setminus\{u\}). \tag{4.3}$$

To estimate the function $\beta^{\star 2}G(t)\bar{F}(o,t)$, we propose a kernel-type estimator $\widehat{H}_n(t)$ defined for $t\in T$ by

$$\widehat{H}_n(t) = \frac{1}{b_n^d|\Lambda_{n,R}|}\sum_{\substack{u,v\in\mathbf{X}\\ v-u\in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u)h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right), \tag{4.4}$$

where $\neq$ over the summation sign means that the sum runs over all pairwise different points $u;v$ in $\mathbf{X}$ and $K:\mathbb{R}^d\to\mathbb{R}$ denotes a smoothing kernel function associated with a sequence $(b_n)_{n\geq 1}$ of bandwidths satisfying the below Condition $K(d,m)$ and Condition $\mathscr{L}$.

Plugging in the above estimators (4.3) and (4.4) and with the convention $c/0=0$ for all real c, we suggest a new edge-corrected non-parametric estimator $\widehat{G}_n(t)$ for $\beta^{\star}G(t)$ by

$$\widehat{G}_n(t) = \frac{\widehat{H}_n(t)}{\widehat{\bar{F}}_n(t)}, \quad \text{for} \quad t\in T. \tag{4.5}$$

To establish strong uniform consistency results for the non-parametric quantity $\widehat{G}_n(t)$ defined by (4.5), we have to impose certain natural restrictions on the kernel function K and the sequence $(b_n)_{n\geq 1}$:

Condition $K(d,m)$**:** The sequence of bandwidths $b_n>0$ for $n\geq 1$, is chosen such that

$$\lim_{n\to\infty} b_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n^d|\Lambda_{n,R}| = \infty.$$

The kernel function $K:\mathbb{R}^d\longrightarrow\mathbb{R}$ is non-negative and bounded with bounded support, and satisfies:

$$\int_{\mathbb{R}^d} K(u)\mathrm{d}u = 1.$$

Let $u=(u_1,\dots,u_d)'$, $u_i\in\mathbb{R}$,

$$\int_{\mathbb{R}^d} u_1^{i_1}\dots u_d^{i_d}K(u_1,\dots,u_d)\mathrm{d}u_1\dots\mathrm{d}u_d = 0, \quad \text{for} \quad 0<\sum_{j=1}^{d} i_j < m.$$

To establish the strong uniform consistency of the kernel-type estimator (4.4) over some compact set $T_0 \subset T$ we need a further smoothness condition on the kernel function.

Condition $\mathscr{L}$: The kernel function K satisfies a Lipschitz condition, i.e. there exists a constant $\eta > 0$ such that

$$\left|K(u) - K(v)\right| \leq \eta ||u - v|| \quad \text{for any} \quad u, v \in T.$$

We also assume additionally that:

Condition $\mathscr{C}(T)$:

$\bar{F}(o,t)$ and $G(t)$ are continuous on any fixed compact set $T_0 \subset T$.

Condition $\mathscr{L}'$:

$$\exists \alpha_1 > 0, \ \ \beta^\star \bar{F}(o,t) \geq \alpha_1 \quad \text{and} \quad \exists \alpha_2 > 0, \ \ \beta^\star G(t) \leq \alpha_2, \quad \forall t \in T_0.$$

The following theorem gives an asymptotic representation for the mean of the kernel-type estimator (4.4).

Theorem 4.1. *Consider a stationary pairwise interaction point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* (3.1) *satisfying condition* (3.2). *Furthermore the kernel function* K *satisfies Condition* $K(d,1)$. *We have*

$$\lim_{n\to\infty} \mathrm{E}\widehat{H}_n(t) = \beta^{\star 2} G(t)\bar{F}(o,t)$$

at any continuity point $t \in T$ *of* $G\bar{F}$.

If Condition $K(d,m)$ *is satisfied and* $G\bar{F}$ *has bounded and continuous partial derivatives of order* m *in an open ball* $B^o(t,\delta)$ *(for some* $\delta > 0$*) for* $t \in \mathring{T}$, *then*

$$\mathrm{E}\widehat{H}_n(t) = \beta^{\star 2} G(t)\bar{F}(o,t) + \mathscr{O}(b_n^m) \quad as \quad n \to \infty. \tag{4.6}$$

The estimator (4.3) turns out to be an unbiased estimator of $\beta^\star \bar{F}(o,t)$ and if $\mathbf{X}$ is ergodic, strongly consistent (the uniform strong consistency) as n tends infinity, since a classical ergodic theorem for spatial point processes obtained in [24]. This implies the following:

Proposition 1. Consider a stationary pairwise interaction point process $\mathbf{X}$ in $\mathbb{R}^d$ with Papangelou conditional intensity (3.1) satisfying condition (3.2) and we assume that Condition $\mathscr{L}'$ is fulfilled. For any fixed compact set $T_0 \subset T$, we have

$$\sup_{t\in T_o} |\widehat{G}_n(t) - \beta^\star G(t)| \longrightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad \text{iff}$$

$$\sup_{t\in T_o} |\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)| \longrightarrow 0 \quad \mathbb{P}\text{-a.s.}.$$

To get a strong uniform consistency of the estimator $\widehat{H}_n(t)$ to the function $\beta^{\star 2}G(t)\bar{F}(o,t)$, we assume that the domain Λ_n is divided into a fixed number of subdomains as follows $\Lambda_n = \cup_{i\in I_n}\Lambda_i$, following Preston [30], Klein [16], Jensen and Künsch [14] we will describe a point process in $\mathbb{R}^d$ as lattice field by means of this decomposition $\Lambda_i = \{\xi \in \mathbb{R}^d; \widetilde{q}(i_j - \frac{1}{2}) \le \xi_j \le \widetilde{q}(i_j + \frac{1}{2}), j = 1,\dots,d\}$ for a fixed number $\widetilde{q} > 0$, $i = (i_1,\dots,i_d)$, and setting $\mathbf{X}_i = \mathbf{X}_{\Lambda_i}$, $i \in \mathbb{Z}^d$, this becomes a Gibbs lattice field. We will consider estimation of $\beta^{\star 2}G(t)\bar{F}(o,t)$ from $\widehat{H}_n(t)$, where $\Lambda_n = \cup_{i\in I_n}\Lambda_i$, and where the process is observed in $\Lambda_{n,R} = \cup_{i\in\widetilde{I}_n}\Lambda_i$, where $\widetilde{I}_n = \{i \in I_n; |i-j| \le 1$, for all $j \in I_n\}$, and the norm is $|j| = \max\{|j_1|,\dots,|j_d|\}$ and assume that $\widetilde{I}_n$ increases towards $\mathbb{Z}^d$ and write

$$\widehat{H}_n(t) = \frac{1}{b_n^d|\Lambda_{n,R}|}\sum_{i\in\widetilde{I}_n}\sum_{\substack{u\in\mathbf{X}_i, v\in\mathbf{X}\\ v-u\in B(o,R)}}^{\neq} h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right). \tag{4.7}$$

To shorten the notation we introduce the random variables

$$Z_i = \sum_{\substack{u,v\in\mathbf{X}\\ v-u\in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_i}(u)h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right) \quad \text{and} \quad \bar{Z}_i = Z_i - \mathrm{E}Z_i$$

for all $i \in \widetilde{I}_n$.

Recall that a Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty}\psi(t) = +\infty$ and $\psi(0) = 0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ such that $E[\psi(|Z|/c)] < +\infty$ for some $c > 0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi = \inf\{\, c > 0; E[\psi(|Z|/c)] \le 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [17]. Let $\theta > 0$. We denote by ψ_θ the exponential Young function defined for any $x \in \mathbb{R}^+$ by

$$\psi_\theta(x) = \exp((x+\xi_\theta)^\theta) - \exp(\xi_\theta^\theta) \quad \text{where} \quad \xi_\theta = ((1-\theta)/\theta)^{1/\theta}\, 1\!\!1\{0 < \theta < 1\}. \tag{4.8}$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i = (i_1,\dots,i_d)$ and $j = (j_1,\dots,j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{lex} j$ means that either $i_1 < j_1$ or for some p in $\{2,3,\dots,d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \le q < p$. Let the sets $\{V_i^k; i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d; j <_{lex} i\},$$

and for $k \geq 2$

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d\,;\, |i-j| \geq k\} \quad \text{where} \quad |i-j| = \max_{1\leq l\leq d} |i_l - j_l|.$$

For any subset I_n of $\mathbb{Z}^d$ define $\mathscr{F}_{I_n} = \sigma(\bar{Z}_i\,;\, i \in I_n)$ and set

$$E_{|k|}(\bar{Z}_i) = E(\bar{Z}_i | \mathscr{F}_{V_i^{|k|}}), \quad k \in V_i^1. \tag{4.9}$$

Denote $\theta(q) = 2q/(2-q)$ for $0 < q < 2$.

Theorem 4.2. *We assume that Conditions $K(d,m)$, $\mathscr{C}(T)$ and $\mathscr{Z}$ are fulfilled. Furthermore, we also assume that $G\bar{F}$ has bounded and continuous partial derivatives of order m on a compact set $T_0 \subset T$. If there exists $0 < q < 2$ such that the centered random variable $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$. Then*

$$\sup_{t\in T_o} \left|\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)\right| = \mathscr{O}_{a.s.}\left(\frac{(\log n)^{1/q}}{(b_n\sqrt{n})^d}\right) + \mathscr{O}(b_n^m) \quad as \quad n \to \infty.$$

Our results also carry through the most important particular case of Orlicz spaces random fields, i.e. For $\psi_p(t) = |t|^p$, for any $t \in \mathbb{R}^+$, the Luxembourg norm is nothing but the $\mathbb{L}^p$-norm.

Theorem 4.3. *We assume that Conditions $K(d,m)$, $\mathscr{C}(T)$ and $\mathscr{Z}$ are fulfilled. Furthermore, we also assume that $G\bar{F}$ has bounded and continuous partial derivatives of order m on a compact set $T_0 \subset T$. If there exists $p > 2$ such that the centered random variable $\bar{Z}_0 \in \mathbb{L}^p$. Assume that $b_n = n^{-q_2}(\log n)^{q_1}$ for some $q_1, q_2 > 0$. Let $a, b \geq 0$ be fixed and if $a(p+d) - d^2/2 - q_2 d > 1 \quad and \quad b(p+d) + q_1 d > 1$. Then*

$$\sup_{t\in T_o} \left|\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)\right| = \mathscr{O}_{a.s.}\left(\frac{n^a(\log n)^b}{(b_n\sqrt{n})^d}\right) + \mathscr{O}(b_n^m) \quad as \quad n \to \infty.$$

5 Proofs

Keep in mind h is given by (4.1) and the function $\bar{F}(o,t)$ defined by (4.2).

5.1 Proof of Theorem 4.1

$\widehat{H}_n(t)$ as a double sum and by means of the GNZ formula (2.2) with

$$h(u,v,\mathbf{X}) = 1\!\!1_{\Lambda_{n,R}}(u)1\!\!1(v-u \in B(o,R))h(u,\mathbf{X})h(v,\mathbf{X})K\left(\frac{v-u-t}{b_n}\right),$$

we get

$$\mathrm{E}\widehat{H}_n(t) = \frac{1}{b_n^d|\Lambda_{n,R}|}\mathrm{E}\sum_{\substack{u,v\in\mathbf{X}\\ v-u\in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u)h(u,\mathbf{X}\setminus\{u,v\})h(v,\mathbf{X}\setminus\{u,v\})K\left(\frac{v-u-t}{b_n}\right)$$
$$= \frac{1}{b_n^d|\Lambda_{n,R}|}\mathrm{E}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u)\,1\!\!1(v-u\in B(o,R))h(u,\mathbf{X})h(v,\mathbf{X})K\left(\frac{v-u-t}{b_n}\right)\lambda_{\beta^\star}(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v.$$

We remember the second order Papangelou conditional intensity by:

$$\lambda_{\beta^\star}(u,v,\mathbf{x}) = \lambda_{\beta^\star}(u,\mathbf{x})\lambda_{\beta^\star}(v,\mathbf{x}\cup\{u\}) \quad \text{for any } u,v\in\mathbb{R}^d \quad \text{and } \mathbf{x}\in N_{lf}.$$

Using the finite range property (3.2) for each function $\lambda_{\beta^\star}(u,\mathbf{x})$ and $\lambda_{\beta^\star}(v,\mathbf{x}\cup\{u\})$, we have

$$\begin{aligned}\lambda_{\beta^\star}(u,\mathbf{X}) &= \lambda_{\beta^\star}(u,\mathbf{X}\cap B(u,R))\\ &= \beta^\star \quad \text{when} \quad \mathbf{X}\cap B(u,R)=\emptyset\end{aligned}$$

and

$$\begin{aligned}\lambda_{\beta^\star}(v,\mathbf{X}\cup\{u\}) &= \lambda_{\beta^\star}(v,(\mathbf{X}\cup\{u\})\cap B(v,R))\\ &= \beta^\star G(v-u) \quad \text{when} \quad \mathbf{X}\cap B(v,R)=\emptyset \quad \text{and} \quad v-u\in B(o,R).\end{aligned}$$

In this way, by stationarity of $\mathbf{X}$, we obtain

$$\mathrm{E}\widehat{H}_n(t) = \frac{\beta^{\star 2}}{b_n^d|\Lambda_{n,R}|}\mathrm{E}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u)\,1\!\!1(v-u\in B(o,R))h(u,\mathbf{X})h(v,\mathbf{X})K\left(\frac{v-u-t}{b_n}\right)G(v-u)\mathrm{d}u\mathrm{d}v$$
$$= \frac{\beta^{\star 2}}{b_n^d|\Lambda_{n,R}|}\int_{\mathbb{R}^{2d}} 1\!\!1_{\Lambda_{n,R}}(u)\,1\!\!1(v-u\in B(o,R))\bar{F}(o,v-u)K\left(\frac{v-u-t}{b_n}\right)G(v-u)\mathrm{d}u\mathrm{d}v$$
$$= \beta^{\star 2}\int_{\mathbb{R}^d} 1\!\!1(b_nz+t\in B(o,R))\bar{F}(o,b_nz+t)K(z)G(b_nz+t)\mathrm{d}z.$$

The continuity of $\bar{F}G$ in t and the boundedness conditions on the kernel function yield the desired result with the first statement of Theorem 4.1 by dominated convergence theorem.

For the results of the second part of Theorem 4.1, we consider $B^0(t,\delta)$ an open ball in $\mathbb{R}^d$, the function $G(t)\bar{F}(o,t)$ has bounded and continuous partial derivatives of order m in $B^o(t,\delta)$ (for some $\delta>0$) for $t\in\mathring{T}$, then for any point z in $\mathbb{R}^d$, there exists $\theta\in(0,1)$, such that by Taylor-Lagrange formula, we get

$$G(t+b_nz) = G(t) + \sum_{k=1}^{m-1}\sum_{|\alpha|=k}\frac{(b_nz)^\alpha}{\alpha!}\frac{\partial^k G(t)}{\partial x^\alpha} + b_n^m R_m(z,t),$$

where $R_m(z,t) = \sum_{|\alpha|=m} \frac{z^\alpha}{\alpha!} \frac{\partial^m G}{\partial x^\alpha}(t + \theta b_n z)$.

$$\bar{F}(o,t+b_n z) = \bar{F}(o,t) + \sum_{k=1}^{m-1} \sum_{|\alpha|=k} \frac{(b_n z)^\alpha}{\alpha!} \frac{\partial^k \bar{F}(o,t)}{\partial x^\alpha} + b_n^m R'_m(z,t),$$

where $R'_m(z,t) = \sum_{|\alpha|=m} \frac{z^\alpha}{\alpha!} \frac{\partial^m \bar{F}}{\partial x^\alpha}(o,t+\theta b_n z)$.

So we multiply two such functions, their product equals the product of their m^{th} Taylor polynomials plus terms involving powers of t higher than m. In other words, to compute the m^{th} Taylor polynomial of a product of two functions, find the product of their Taylor polynomials, ignoring powers of t higher than m. So we denote this product by $T(t)(b_n z)^\alpha$, then we have

$$\begin{aligned}
\mathrm{E}\widehat{H}_n(t) &= \beta^{\star 2} G(t)\bar{F}(o,t) \int_{\mathbb{R}^d} K(z)\mathrm{d}z \\
&\quad + \beta^{\star 2} T(t) b_n^\alpha \int_{\mathbb{R}^d} z^\alpha K(z)\mathrm{d}z \\
&\quad + \mathscr{O}(b_n^m) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Together with Condition $K(d,m)$, we get the asserted rate of convergence. □

5.2 Proof of Theorem 4.2 and Theorem 4.3

To establish rates of the uniform $\mathbb{P}$- a.s. convergence for the estimator (4.7), we apply a triangle inequality decomposition allows for

$$\sup_{t\in T_0} \left|\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)\right| \leq \sup_{t\in T_0} \left|\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)\right| + \sup_{t\in T_0} \left|\mathrm{E}\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)\right|. \tag{5.1}$$

Let $(r_n)_{n\geq 1}$ be sequence of positive numbers going to zero. Following Carbon and al. [4], the compact set T_0 can be covered by v_n cubes T_k having sides of length $l_n = r_n b_n^{d+1}$ and center at c_k. Clearly there exists $c > 0$, such that $v_n \leq c/l_n^d$. Define

$$A_1 = \max_{1\leq k\leq v_n} \sup_{t\in T_k} \left|\widehat{H}_n(t) - \widehat{H}_n(c_k)\right|$$

$$A_2 = \max_{1\leq k\leq v_n} \sup_{t\in T_k} \left|\mathrm{E}\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(c_k)\right|$$

$$A_3 = \max_{1\leq k\leq v_n} \left|\widehat{H}_n(c_k) - \mathrm{E}\widehat{H}_n(c_k)\right|$$

then

$$\sup_{t\in T_o} |\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)| \leq A_1 + A_2 + A_3. \tag{5.2}$$

We study the following lemmas which significantly improve the desired result.

Lemma 1. For j=1,2, we have

$$A_j = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n \to \infty.$$

Proof. For any $t \in T_k$, by Condition $\mathscr{X}$, we derive that there exists constant $\eta > 0$ such that as $n \to \infty$

$$\begin{aligned}
&\left|\widehat{H}_n(t) - \widehat{H}_n(c_k)\right| \\
&\leq \frac{1}{b_n^{d+1}} \eta \left\|t - c_k\right\| \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right| \\
&\leq \frac{1}{b_n^{d+1}} \eta r_n b_n^{d+1} \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right| \\
&\leq \eta r_n \left| \frac{1}{|\Lambda_{n,R}|} \sum_{\substack{u,v \in \mathbf{X} \\ v-u \in B(o,R)}}^{\neq} 1\!\!1_{\Lambda_{n,R}}(u) h(u, \mathbf{X} \setminus \{u,v\}) h(v, \mathbf{X} \setminus \{u,v\}) \right|.
\end{aligned}$$

From the last inequality and the Nguyen and Zessin [24] ergodic theorem, it easily follows

$$A_1 = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n \to \infty.$$

Then one gets with the same arguments as before and by L^1-version of the ergodic theorem of Nguyen and Zessin [24]:

$$A_2 = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n \to \infty.$$

The proof of Lemma 1 is complete. □

We now concentrating on the stochastic part.

Lemma 2. Assume that for $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$, for some $0 < q < 2$ and $r_n = (\log n)^{1/q}/(b_n\sqrt{n})^d$ holds. Then we have,

$$A_3 = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n \to \infty.$$

Proof. To establish the proof of Lemma 2 we need the following result. Consider the following assumption:

$$\exists q \in]0,2[\quad \exists c > 0 \quad \mathrm{E}[\exp(c|X_0|^{\theta(q)})] < +\infty \tag{5.3}$$

$\theta(q) = 2q/(2-q)$.

Theorem 5.1. *(El Machkouri, [19]). Let $(X_i)_{i\in\mathbb{Z}^d}$ be a zero mean stationary real random field which satisfies the assumption (5.3) for some $0<q<2$. There exists a positive universal constant $M_1(q)>0$ depending only on q such that for any family $(a_i)_{i\in\mathbb{Z}^d}$ of real numbers and any finite subset Γ in $\mathbb{Z}^d$,*

$$\Big\|\sum_{i\in\Gamma} a_i X_i\Big\|_{\psi_q} \le M_1(q)\Big(\sum_{i\in\Gamma}|a_i| b_{i,q}(X)\Big)^{1/2}$$

where

$$b_{i,q}(X) = |a_i|\|X_0\|^2_{\psi_{\theta(q)}} + \sum_{k\in V_0^1}|a_{k+i}|\left\|\sqrt{|X_k E_{|k|}(X_0)|}\right\|^2_{\psi_{\theta(q)}}.$$

Now, in the sequel, the letter C denotes any generic positive constant. We consider the exponential Young function defined by (4.8). Let $\varepsilon>0$ and $t\in T_0$ be fixed

$$\begin{aligned}\mathbb{P}\left(|\widehat{H}_n(t)-\mathrm{E}\widehat{H}_n(t)|>\varepsilon r_n\right) &= \mathbb{P}\left(|\sum_{i\in\widetilde{I}_n}\bar{Z}_i|>\varepsilon r_n(nb_n)^d\right)\\ &\le \exp\left[-\left(\frac{\varepsilon\, r_n(nb_n)^d}{||\sum_{i\in\widetilde{I}_n}\bar{Z}_i||_{\psi_q}}+\xi_q\right)^q\right]\mathrm{E}\exp\left[\left(\frac{|\sum_{i\in\widetilde{I}_n}\bar{Z}_i|}{||\sum_{i\in\widetilde{I}_n}\bar{Z}_i||_{\psi_q}}+\xi_q\right)^q\right].\end{aligned}$$

Therefore, we assume that there exists a real $0<q<2$, such that $\bar{Z}_0\in\mathbb{L}_{\psi_{\theta(q)}}$. Applying Theorem 5.1 to the sequence $\bar{Z}_i$, $i\in\widetilde{I}_n$, we have

$$\begin{aligned}\mathbb{P}\left(|\widehat{H}_n(t)-\mathrm{E}\widehat{H}_n(t)|>\varepsilon r_n\right) &\le \mathbb{P}\left(|\sum_{i\in\widetilde{I}_n}\bar{Z}_i|>\varepsilon r_n(nb_n)^d\right)\\ &\le (1+e^{\xi_q^q})\exp\left[-\left(\frac{\varepsilon\, r_n(nb_n)^d}{M(\sum_{i\in\widetilde{I}_n}b_{i,q}(\bar{Z}))^{1/2}}+\xi_q\right)^q\right]\end{aligned}$$

where

$$b_{i,q}(\bar{Z}) = \|\bar{Z}_0\|^2_{\psi_{\theta(q)}} + \sum_{k\in V_0^1}\left\|\sqrt{|\bar{Z}_k E_{|k|}(\bar{Z}_0)|}\right\|^2_{\psi_{\theta(q)}},$$

M is a positive constant depending only on q.

Note that, the random field $\bar{Z}_i = Z_i-\mathrm{E}Z_i$ depends on $\mathbf{X}_{\Lambda_i}$, for any $i\in\widetilde{I}_n$, i.e. $|i-j|\le 1$, for $j\in I_n$ only. Then we deduce from the construction of $\mathscr{F}_{V_i^{|k|}}$ defined by (4.9), that the random field Z_i is independent of $\mathscr{F}_{V_i^{|k|}}$, and we conclude that the conditional means of $\bar{Z}_i$ given $\mathscr{F}_{V_i^{|k|}}$ is zero, i.e. $E_{|k|}(\bar{Z}_0)=0$. Then we derive that $b_{i,q}(\bar{Z})=\|\bar{Z}_0\|^2_{\psi_{\theta(q)}}$. It results for $\bar{Z}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$ that there exists constant $C > 0$ and so if $r_n = (\log n)^{1/q}/(b_n\sqrt{n})^d$, such that

$$\sup_{t\in T_0} \mathbb{P}\left(|\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)| > \varepsilon r_n\right) \leq (1+e^{\xi_q^q}) \exp\left[-\left(\frac{\varepsilon\, r_n(\sqrt{n}b_n)^d}{C} + \xi_q\right)^q\right]$$
$$\leq (1+e^{\xi_q^q}) \exp\left[-\frac{\varepsilon^q \log n}{C^q}\right]. \tag{5.4}$$

On the other hand, we have

$$\mathbb{P}\left(|A_3| > \varepsilon r_n\right) \leq v_n \sup_{t\in T_0} \mathbb{P}\left(|\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)| > \varepsilon r_n\right).$$

Using (5.4), choosing ε sufficiently large, therefore, it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n\to\infty} |A_3| > \varepsilon r_n) = 0.$$

□

Now, we will accomplish the proof of Theorem 4.3.

Lemma 3. Assume $\bar{Z}_0 \in \mathbb{L}^p$ for some $p > 2$ and $b_n = n^{-q_2}(\log n)^{q_1}$ for some $q_1, q_2 > 0$. Let $a, b \geq 0$ be fixed and denote $r_n = n^a(\log n)^b/(b_n\sqrt{n})^d$. If $a(p+d) - d^2/2 - q_2 d > 1$ and $b(p+d) + q_1 d > 1$. Then we have

$$A_3 = \mathcal{O}_{a.s}(r_n) \quad \text{as} \quad n \to \infty.$$

Proof. Let $p > 2$ be fixed, such that $\bar{Z}_0 \in \mathbb{L}^p$ and for any $\varepsilon > 0$,

$$\mathbb{P}(|\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)| > \varepsilon r_n) = \mathbb{P}\left(|\sum_{i\in\widetilde{I}_n} \bar{Z}_i| > \varepsilon r_n (nb_n)^d\right)$$
$$\leq \frac{\varepsilon^{-p}\mathrm{E}|\sum_{i\in\widetilde{I}_n} \bar{Z}_i|^p}{r_n^p (nb_n)^{pd}}$$
$$\leq \frac{\varepsilon^{-p}}{r_n^p (nb_n)^{pd}} \left(2p \sum_{i\in\widetilde{I}_n} c_i(\bar{Z})\right)^{p/2}.$$

The last inequality follows from Dedecker [7], where $c_i(\bar{Z}) = \|\bar{Z}_i\|_p^2 + \sum_{k\in V_i^1} \|\bar{Z}_k E_{|k-i|}(\bar{Z}_i)\|_{\frac{p}{2}}$. The conditional means $\mathrm{E}_{|k-i|}(\bar{Z}_i)$ is zero, then we have $c_i(\bar{Z}) = \|\bar{Z}_i\|_p^2$ and with the stationarity of $\mathbf{X}$, we derive that there exists $C > 0$ such that

$$\mathbb{P}\left(|A_3| > \varepsilon r_n\right) \leq v_n \sup_{t\in T_0} \mathbb{P}(|\widehat{H}_n(t) - \mathrm{E}\widehat{H}_n(t)| > \varepsilon r_n)$$
$$\leq v_n \frac{C\varepsilon^{-p}}{r_n^p (b_n\sqrt{n})^{pd}}.$$

As $v_n \le c/l_n^d$ and $l_n = r_n b_n^{1+d}$, therefore for $r_n = n^a(\log n)^b/(b_n\sqrt{n})^d$, it follows as $n \to \infty$,

$$\begin{aligned}\mathbb{P}(|A_3| > \varepsilon r_n) &\le \frac{C\varepsilon^{-p}}{n^{a(p+d)-d^2/2}(\log n)^{b(p+d)} b_n^d} \\ &\le \frac{C\varepsilon^{-p}}{n^{a(p+d)-d^2/2-q_2 d}(\log n)^{b(p+d)+q_1 d}}.\end{aligned}$$

For $a(p+d)-d^2/2-q_2d > 1$ and $b(p+d)+q_1d > 1$, we have $\sum_{n\ge 1}\mathbb{P}(|A_3| > \varepsilon r_n) < \infty$.

The proof of Lemma 3 is complete. □

By strengthening the uniform continuity assumption (Condition $\mathscr{C}(T)$) of $G(t)\bar{F}(o,t)$, one hand, by Theorem 4.1, we have

$$\sup_{t\in T_o} |\mathrm{E}\widehat{H}_n(t) - \beta^{\star 2}G(t)\bar{F}(o,t)| = \mathscr{O}(b_n^m).$$

We conclude the proof of Theorem 4.2 and Theorem 4.3 by combining the inequality (5.1) and the inequality (5.2). □

5.3 Proof of Proposition 1

We consider the decomposition

$$\widehat{G}_n(t) - \beta^\star G(t) = \frac{\widehat{H}_n(t) - \beta^{\star 2}G(t)\bar{F}(o,t)}{\widehat{\bar{F}}_n(t)} + \beta^\star G(t)\frac{\widehat{\bar{F}}_n(t) - \beta^\star \bar{F}(o,t)}{\widehat{\bar{F}}_n(t)}. \tag{5.5}$$

Additionally, assume that $\mathbb{P}$ is ergodic, the ergodic theorem (Nguyen and Zessin [24]) immediately gives, as $n \to \infty$,

$$\left|\widehat{\bar{F}}_n(t) - \beta^\star \bar{F}(o,t)\right| \xrightarrow{a.s.} 0. \tag{5.6}$$

Note that there exists at least one stationary Gibbs measure. If this measure is unique, it is ergodic. Otherwise, it can be represented as a mixture of ergodic measures (see [12], Theorem 14.10).

Now, using the monotony of functions $\bar{F}$ and $\widehat{\bar{F}}_n$, we can approach the functions $\beta^\star\bar{F}$ and $\widehat{\bar{F}}_n$ by their values in a finite number of points. Bringing this remark and the result (5.6), we have as $n \to \infty$,

$$\sup_{t\in T_o}\left|\widehat{\bar{F}}_n(t) - \beta^\star \bar{F}(o,t)\right| \xrightarrow{a.s.} 0. \tag{5.7}$$

By (5.6) and by Condition $\mathscr{L}'$, we obtain at any point $t \in T_0$

$$\widehat{\bar{F}}_n(t) > \alpha_1, \quad \mathbb{P}-\text{a.s.},$$

furthermore, by relation (5.5), we also see that

$$\left|\widehat{G}_n(t) - \beta^\star G(t)\right| \le {\alpha_1}^{-1}\left|\widehat{H}_n(t) - \beta^{\star 2} G(t)\bar{F}(o,t)\right| + \alpha_2 \alpha_1^{-1}\left|\widehat{\bar{F}}_n(t) - \beta^\star \bar{F}(o,t)\right|. \quad (5.8)$$

Together with Theorem 4.2, the expression (5.7) and the expression (5.8), we complete the proof. □

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